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GENERAL STABILITY OF TWO-DIMENSIONAL SLOPES BASED ON SARMA'S METHOD

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SUMMARY

Sarma's method of taking a fictitious accerelation as a measure of safety is used to study a two-dimensional slope. The slope is divided into arbitrary slices. The relations among forces acting on the slices and Sarma's acceleration are assumed to be linear. Consequently, a general analytical expression of Sarma's acceleration is derived by means of Cramer's rule. Furthermore, for four commonly used slice methods with simple relations among the forces on each slice and Sarma's acceleration, a general closed-form solution of Sarma's acceleration is given. An example is calculated and the results agree well with those of Hoek. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: Sarma's method; two-dimensional slope; stability

1. INTRODUCTION

In conventional slope stability analysis, iterative computation is necessary to get the factor of safety. The iteration increases the computational effort and brings about the problem of convergence. For this reason, Sarma proposed a fictitious horizontal acceleration K_c (see Figure 1) as a safety measure of a two-dimensional slope.[1] When $K_c > 0$, the slope is stable; when $K_c = 0$, it is in a critical state; when $K_c < 0$, it is unstable. Examples have been computed for vertical slices.[1] In Sarma's examples, the shear force distributions between neighbouring slices have been assumed. It has been pointed out that the critical acceleration K_c has an approximately linear relationship with the factor of safety FS.[2]

Sarma's method has a few adantages, e.g. it avoids problems due to the iterative computation; it is simple but gives exact solutions,[2] it works for arbitrary slices.[3,4]

In this paper, a two-dimensional slope is further analysed based on Sarma's method. Study is focused on five commonly used slice methods, simply called method-1 to method-5. Method-1 is a generalized Bishop's method applicable to arbitrary sliding surfaces and slices, extending the original one which is limited to circular failure surfaces with vertical slice boundaries.[5] In this method, the resultant of the side forces need not be horizontal. In method-2, the total force on each slice side is assumed to be parallel to the base of the slice; In method-3, all the sides and bases

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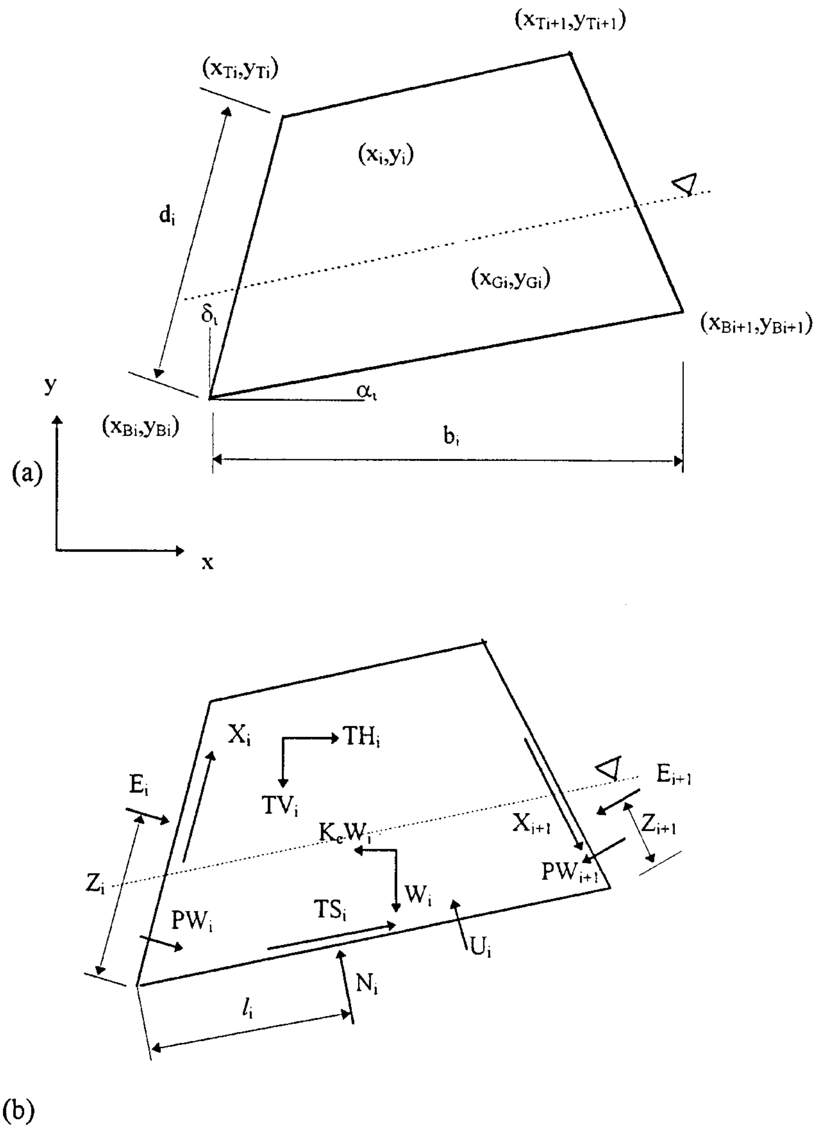

Figure 1. Definition of geometry and forces acting on the *i*th slice. (a) Geometry of the *i*th slice. (b) Forces acting on the *i*th slice

of slices are in the state of limiting equilibrium;[3,4] In method-4, a distribution of the shear force among slices, which is determined from the equilibrium of the slices and the whole slope (for details see Appendix II), is assumed.[6] When a certain distribution is given, its amplitude may be determined by a set of equilibrium equations of slices. In the following analysis, a distribution similar to that used by Sarma[1] is assumed. Method-5 is Janbu's method.[7]

A linear equation group can be formed via Sarma's method, which includes equilibrium equations of slices and constraint equations (also called supplemental equations). In the equation

group, K_c is one of the unknown quantities. Once the coefficients of the equations are known, K_c may be expressed formally by the well-known Cramer's rule. The expression has the feature of generality, i.e. it may conform to different slice methods. For different slice methods, the equilibrium equations of slices are the same. The constraint equations include the limiting conditions of slice bases together with other equations based on proper assumptions. The limiting conditions of slice bases are obviously independent of slice methods. However, the equations based on proper assumptions vary with slice methods, which may be regarded as key constraint equations. The mathematical distinction between any slice method depends mainly on their key constraint equations. Different slope analyses can be unified by modifying the key constraint equations.

It is found that, although being formed under different assumptions, the key constraint equations of method-1 to method-4 may all be written as linear relations among Sarma's acceleration, the shear force, and the effective normal force between neighbouring slices. Therefore, a general analytical solution of Sarma's acceleration can be derived by solving a linear equation group. It is consistent with Sarma's formula[3,4] in form, but extends its applicable range. Consequently, the computation of a two-dimensional slope under the four slice methods, method-1–method-4, can be performed in a single program. It is simple and effective for engineering applications. In addition, it is easy to compare results of different slice methods.

The method proposed here is used to re-compute an example which has been computed by Hoek.[4] The results are found to agree well with Hoek's.

2. GENERAL ANALYTICAL EXPRESSION OF SARMA'S ACCELERATION

For simplicity, many of Hoek's[4] and Sarma's[1] notations are followed in this paper. Let a two-dimensional slope be divided into n slices with arbitrary shapes. The geometry parameters of and forces on the ith slice are shown in figure 1(a) and 1(b), respectively, where W_i is the slice weight, (X_{G_i}, Y_{G_i}) is the centre of gravity; PW_i, PW_{i+1} and U_i are forces due to water pressures; E_i, E_{i+1} and N_i are separately resultant normal forces acting on the sides and base of the slice; X_i, X_{i+1} and TS_i are shear forces on the sides and base; TH_i and TV_i are, respectively, the horizontal and vertical external forces, acting at the point (x_i, y_i); d_i is length of the ith side; $(x_{\mathrm{B}i}, y_{\mathrm{B}i})$, $(x_{\mathrm{B}i+1}, y_{\mathrm{B}i+1})$, $(x_{\mathrm{T}i}, y_{\mathrm{T}i})$ and $(x_{\mathrm{T}i+1}, y_{\mathrm{T}i+1})$ are four corners of the slice; α_i is the angle made by the slice base with x-axis; δ_i is the angle made by the slice side with y-axis. Suppose Sarma's critical acceleration is K_c. There is a horizontal inertia force $K_c W_i$ acting down-slope. Since the value of K_c reflects the reserved strength of the slope, it may be taken as a measure of safety.

The force equilibrium equations of the ith slice in Figure 1 along the x- and y-axis are separately

$$E_i \cos\delta_i - E_{i+1}\cos\delta_{i+1} + X_i \sin\delta_i - X_{i+1}\sin\delta_{i+1} - N_i \sin\alpha_i + \mathrm{TS}_i \cos\alpha_i - W_i K_c = -\mathrm{TH}_i$$

and

$$-E_i \sin\delta_i + E_{i+1}\sin\delta_{i+1} + X_i \cos\delta_i - X_{i+1}\cos\delta_{i+1} + N_i \cos\alpha_i + \mathrm{TS}_i \sin\alpha_i = \mathrm{TV}_i + W_i \quad (1)$$

For the first and the last slice, since $E_1 = 0$, $X_1 = 0$ and $E_{n+1} = 0$, $\mathrm{X}_{n+1} = 0$, equations (1) becomes

$$-E_2 \cos\delta_2 - X_2 \sin\delta_2 - N_1 \sin\alpha_1 + \mathrm{TS}_1 \cos\alpha_1 - W_1 K_c = -\mathrm{TH}_1$$
$$E_2 \sin\delta_2 - X_2 \cos\delta_2 + N_1 \cos\alpha_1 + \mathrm{TS}_1 \sin\alpha_1 = \mathrm{TV}_1 + W_1 \quad (2)$$

and

$$E_n \cos\delta_n + X_n \sin\delta_n - N_n \sin\alpha_n + \mathrm{TS}_n \cos\alpha_n - W_n K_c = -\mathrm{TH}_n$$
$$-E_n \sin\delta_n + X_n \cos\delta_n + N_n \cos\alpha_n + \mathrm{TS}_n \sin\alpha_n = \mathrm{TV}_n + W_n \quad (3)$$

In many cases, the moment equilibrium of slices is not considered since the acting points of resultant normal forces on the slice sides and bases are difficult to determine, unless locations of these points are empirically assumed, as is done in Janbu's method.

The limiting condition for the ith base is

$$\mathrm{TS}_i = c_{\mathrm{B}i} b_i / \cos\alpha_i + (N_i - U_i)\tan\varphi_{\mathrm{B}i} \quad (4)$$

where $b_i = x_{\mathrm{B}i+1} - x_{\mathrm{B}i}$; $\varphi_{\mathrm{B}i}$, $c_{\mathrm{B}i}$ and $b_i/\cos\alpha_i$ are effective shear strength parameters and length of the ith base, respectively.

Equations (1) are valid for $2 \leqslant i \leqslant n-1$, while equation (4) applies for $1 \leqslant i \leqslant n$. Thus there are totally $2n$ equilibrium equations in equations (1)–(3). Meanwhile, there are n constraints, or supplemental equations in equation (4). However, the above equations are insufficient to solve the $(4n-1)$ unknown quantities in them. Thus $(n-1)$ more supplemental equations or the key constraint equations have to be introduced based on proper assumptions for various slice methods. Details of these supplemental equations for method-1–method-5 are given in the next two sections.

Let the unknown quantities be expressed by a vector of the $(4n-1)$th rank, i.e. $\mathbf{B} = (E_2, \ldots, E_n, X_2, \ldots, X_n, N_1, \ldots, N_n, \mathrm{TS}_1, \ldots, \mathrm{TS}_n, K_c)^{\mathrm{T}}$; A linear equation group may be established from the equilibrium and the supplemental equations:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \quad (5)$$

where $\mathbf{A} = (a_{ij})_{(4n-1)(4n-1)}$ is a matrix of the $(4n-1)\times(4n-1)$th rank and $\mathbf{C} = (c_i)_{(4n-1)1}$ is a vector of the $(4n-1)$th rank. In equations (5), the first $2n$ equations are the force equilibrium ones, in which the $(2i-1)$th and the $2i$th are those of the ith slice along the x- and y-axis; the $(2n+1)$th to the $3n$th equations are the limiting conditions of the slice bases; and the $(3n+1)$th to the $(4n-1)$th equataions are the key constraint equations. Elements of the first $3n$ lines of the matrix $\mathbf{A}$ and the vector $\mathbf{C}$ are given in Appendix I. Elements of the last $(n-1)$ lines of the matrix $\mathbf{A}$ and the vector $\mathbf{C}$, $a_{(3n+i),j}$ and $c_{3n+i} (i = 1, \ldots, n-1, j = 1, \ldots, 4n-1)$, rely on certain assumptions of various slice methods.

From equations (5), $\mathbf{B} = \mathbf{A}^{-1} \cdot \mathbf{C}$. Especially, according to Cramer's rule,[8] Sarma's acceleration can be written as

$$K_c = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} \quad (6)$$

where $\mathbf{A}_1$ is a matrix gained by replacing the $(4n-1)$th column of the matrix $\mathbf{A}$ with the vector $\mathbf{C}$; and '| |' denotes determinant.

The solution of equations (5) should satisfy the condition that the effective normal stresses at the slice sides and bases are not negative, i.e.

$$\begin{aligned} &(N_i - U_i)\cos\alpha_i / b_i \geqslant 0, \\ &(E_i - \mathrm{PW}_i)/d_i \geqslant 0, \end{aligned} \quad (i = 1, \ldots, n) \quad (7)$$

Sometimes it is required that the solution does not violate the moment equilibrium of the slices. Take the moments of all the forces acting on the ith slice with respect to its lower-left corner (x_{Bi}, y_{Bi}) as in Figure 1.

$$N_i l_i - X_{i+1} b_i \cos(\alpha_i + \delta_{i+1})/\cos\alpha_i - E_i Z_i + E_{i+1}[Z_{i+1} + b_i \sin(\alpha_i + \delta_{i+1})/\cos\alpha_i]$$
$$- W_i(x_{Gi} - x_{Bi}) + K_c W_i(y_{Gi} - y_{Bi}) - \mathrm{TV}_i(x_i - x_{Gi}) + \mathrm{TH}_i(y_i - y_{Gi}) = 0 \quad (8)$$

Here l_i, Z_i and Z_{i+1} determine, respectively, the acting points of N_i, E_i and E_{i+1}. Since $Z_1 = 0$, when the value of l_i is assumed empirically, $Z_2, \dots, Z_{n-1}$ and Z_n can be solved by recurrence formulas. Note that we must have

$$0 \leqslant Z_i \leqslant d_i, \quad i = 2, \dots, n \quad (9)$$

Let $\mathrm{FS}_{Li} = [(E_i - \mathrm{PW}_i)\tan\varphi_{si} + c_{si} d_i]/X_i$ be the local factor of safety on the ith side, where φ_{Si} and c_{s_i} are shear strength parameters. A reasonable solution should satisfy

$$\mathrm{FS}_{Li} \geqslant 1 \quad (i = 2, 3, \dots, n) \quad (10)$$

3. GENERAL CLOSED-FORM EXPRESSION OF SARMA'S ACCELERATION FOR FOUR SLICE METHODS

In equations (5), to comprise method-1–method-4, the $(n-1)$ key constraint equations are supposed to have the following form:

$$X_{i+1} = (E_{i+1} - \mathrm{PW}_{i+1})\tan\beta_{i+1} + D_{i+1} + G_{i+1}K_c \quad (i = 1, \dots, n-1) \quad (11)$$

where β_{i+1}, D_{i+1} and G_{i+1} are constants. Therefore, elements of the last $(n-1)$ lines of the matrix **A** and the vector **C** are

$$\begin{aligned} &a_{(3n+i),i} = \tan\beta_{i+1},\ a_{(3n+i),(n+i-1)} = -1;\ a_{(3n+i),(4n-1)} = G_{i+1} \\ &c_{(3n+i)} = \mathrm{PW}_{i+1}\tan\beta_{i+1} - D_{i+1};\ a_{(3n+i),j} = 0,\ 1 \leqslant j \leqslant 4n-2\ \&\ j \neq i, n+i-1 \end{aligned} \quad (12)$$

From a derivation similar to Sarma's,[3]

$$K_c = \frac{\sum_{i=1}^{n}(a_i \prod_{j=i+1}^{n} e_j)}{\sum_{i=1}^{n}(p_i \prod_{j=i+1}^{n} e_j)} \quad (13)$$

where

$$\begin{aligned} a_i &= Q_i[S_i \sin(\delta_i - \varphi_{Bi} + \alpha_i) + R_i \cos\varphi_{Bi} - S_{i+1}\sin(\delta_{i+1} - \varphi_{Bi} + \alpha_i) \\ &\quad + \mathrm{TH}_i \cos(\varphi_{Bi} - \alpha_i) + (W_i + \mathrm{TV}_i)\sin(\varphi_{Bi} - \alpha_i)] \\ Q_i &= \cos\beta_{i+1}/\cos(\varphi_{Bi} - \alpha_i + \beta_{i+1} - \delta_{i+1}) \\ e_i &= Q_i \cos(\beta_i - \delta_i + \varphi_{Bi} - \alpha_i)/\cos\beta_i \\ p_i &= Q_i[W_i \cos(\varphi_{Bi} - \alpha_i) + G_{i+1}\sin(\delta_{i+1} + \alpha_i - \varphi_{Bi}) - G_i \sin(\delta_i + \alpha_i - \varphi_{Bi})] \\ S_i &= D_i - \mathrm{PW}_i \tan\beta_i \\ R_i &= c_{Bi} b_i/\cos\alpha_i - U_i \tan\varphi_{Bi} \end{aligned} \quad (14)$$

The coefficients in equation (11) or (12) can be modified to suit method-1–method-4 separately, e.g. for method-1, $\beta_{i+1} = D_{i+1} = G_{i+1} = 0$; for method-2, $\beta_{i+1} = \alpha_{i+1} + \delta_{i+1}$ and $D_{i+1} = G_{i+1} = 0$; method-3 requires that $\beta_{i+1} = \varphi_{si+1}$, $D_{i+1} = C_{s_{i+1}} d_{i+1}$ and $G_{i+1} = 0$, which leads to Sarma's formula.[3,4] It may be shown that, for method-4, $\beta_{i+1} = 0$, $D_{i+1} = (V_2/V_1)F_{i+1}$ and $G_{i+1} = (-V_3/V_1)F_{i+1}$ (for details see Appendix II). Here F_{i+1} ($i = 1, \ldots, n-1$) are pre-valued, $F_1 = F_{n+1} = 0$.

$$\begin{aligned}
V_1 &= \sum [\bar{x}_i - x_G) - (\bar{y}_i - y_G)\tan(\varphi_{Bi} - \alpha_i)] \\
&\quad \times \cos(\varphi_{Bi} - \alpha_i)\sec(\varphi_{Bi} - \alpha_i - \delta_i)(F_{i+1} - F_i) \\
V_2 &= -\sum \{W_i(\bar{x}_i - x_G) + P_i(\bar{y}_i - y_G) \\
&\quad + [(\bar{x}_i - x_G) - (\bar{y}_i - y_G)\tan(\varphi_{Bi} - \alpha_i)]\cos(\varphi_{Bi} - \alpha_i) \\
&\quad \times \sec(\varphi_{Bi} - \alpha_i - \delta_i) P_i \sin\delta_i\} \\
V_3 &= \sum [(\bar{x}_i - x_G) - (\bar{y}_i - y_G)\tan(\varphi_{Bi} - \alpha_i)]\cos(\varphi_{Bi} - \alpha_i) \\
&\quad \times \sec(\varphi_{Bi} - \alpha_i - \delta_i) W_i \sin\delta_i \\
\bar{x}_i &= \tfrac{1}{2}(x_{B_{i+1}} + x_{B_i}); \qquad \bar{y}_i = \tfrac{1}{2}(y_{B_{i+1}} + y_{Bi}) \\
P_i &= (U_i \sin\varphi_{Bi} - C_{Bi} b_i \cos\varphi_{Bi} \sec\alpha_i)\sec(\varphi_{Bi} - \alpha_i) \\
&\quad - (\mathrm{TV}_i + W_i)\tan(\varphi_{Bi} - \alpha_i) - \mathrm{TH}_i,
\end{aligned} \tag{15}$$

where (x_G, y_G) is the centre of gravity of the whole slope. For homogeneous slopes (see Appendix III),

$$F_i = (K'_i - R_{ui})\frac{\gamma_r d_i^2}{2}\cos\delta_i \tan\varphi_{si} + c_{si} d_i \tag{16}$$

where γ_r is unit weight of the slice material; $i = 2, \ldots, n$,

$$\begin{aligned}
R_{ui} &= \frac{U_i}{W_i \sec\alpha_i} \\
K'_i &= \frac{1}{1 + d'_{1i}}\left[d'_{1i}(1 + d_{1i}) + \frac{2(d'_{1i} d_{2i} + d'_{2i})}{\gamma_r d_i \cos\delta_i} + R_{ui}(d'_{1i} d_{3i} + d'_{3i}))\right] \\
d_{1i} &= \frac{1 - \sin\phi_i \sin\varphi_{si}}{1 + \sin\phi_i \sin\varphi_{si}};\ d_{2i} = -\frac{2c_{si}\cos\phi_{si}\sin\phi_i}{1 + \sin\varphi_{si}\sin\phi_i};\ d_{3i} = \frac{2\sin\varphi_{si}\sin\phi_i}{1 + \sin\varphi_{si}\sin\phi_i} \\
d'_{1i} &= \frac{1 - \sin\phi'_i \sin\varphi_{si}}{1 + \sin\phi'_i \sin\varphi_{si}};\ d'_{2i} = -\frac{2c_{si}\cos\varphi_{si}\sin\phi'_i}{1 + \sin\varphi_{si}\sin\phi'_i};\ d'_{3i} = \frac{2\sin\varphi_{si}\sin\phi'_i}{1 + \sin\varphi_{si}\sin\phi'_i} \\
\phi_i &= 2\alpha_i - \varphi_{si},\ \phi'_i = 2(\alpha_i + \delta_i) - \varphi_{si}
\end{aligned} \tag{17}$$

The formulas derived by Sarma[1] for vertical slices constitute a special case of equations (15)–(17).

4. SARMA'S ACCELERATION FOR JANBU'S METHOD

In Figure 1, let the moment equilibrium equations of the first $(n-1)$ slices, i.e. equation (8) for $i = 1, \dots, (n-1)$, be the key constraint equations of equations (5). Consequently elements of the last $(n-1)$ lines of the matrix **A** and the vector **C** are

$$
\begin{aligned}
a_{(3n+i),(i-1)} &= -Z_i;\ a_{(3n+i),i} = Z_{i+1} + b_i \sin(\alpha_i + \delta_{i+1})/\cos\alpha_i \\
a_{(3n+i),(n+i-1)} &= -b_i \cos(\alpha_i + \delta_{i+1})/\cos\alpha_i \\
a_{(3n+i),(2n-2+i)} &= l_i;\ a_{(3n+i),(4n-1)} = W_i(y_{Gi} - y_{Bi}) \\
c_{(3n+i)} &= W_i(x_{Gi} - x_{Bi}) + \mathrm{TV}_i(x_i - x_{Gi}) - \mathrm{TH}_i(y_i - y_{Gi}) \\
a_{(3n+i),j} &= 0, \quad (1 \leqslant j \leqslant 4n-2\ \&\ j \neq i-1, i, n+i-1, 2n+i-2)
\end{aligned}
\tag{18}
$$

where the values of Z_i and l_i, which denote the acting points of E_i and N_i, respectively, are assumed $(i = 1, 2, \dots, n-1)$. Therefore, Sarma's acceleration for Janbu's method can be obtained. Moment equilibrium of the nth slice gives

$$N_n l_n - E_n Z_n - W_n(x_{Gn} - x_{Bn}) + K_c W_n(y_{Gn} - y_{Bn}) - \mathrm{TV}_n(x_n - x_{Gn}) + \mathrm{TH}_n(y_n - y_{Gn}) = 0 \tag{19}$$

For reasonable values of Z_i and l_i, the value of l_n given by equation (19) should satisfy $0 \leqslant l_n \leqslant b_n/\cos\alpha_n$.

5. EXAMPLE AND DISCUSSION

For comparison, an example which has been computed by Hoek[4] is re-considered in this paper. As shown in Figure 2, it is an open-pit coal mine slope, where the dotted line denotes the underground water. Parameters concerned and Hoek's results using method-3 are listed in Table I, where (x_{Wi}, y_{Wi}) is the intersection of the water level with the ith slice side, necessary in computing the force PW_i. Results obtained in this paper using the five-slice methods are listed in Table II. Good agreement can be found between Hoek's results, i.e. Sarma's acceleration K_c and the factor of safety FS, and the present calculation under method-3. Here the factor of safety has been computed via the method suggested by Hoek,[4] dividing the shear strength of each slice, $\tan\varphi_{Si}$, c_{si} and $\tan\varphi_{Bi}$, c_{Bi}, simultaneously by a factor F. Consequently, the computed Sarma's acceleration varies, which may be called the intermediate Sarms's acceleration, denoted by K'_c. When K'_c equals zero, the corresponding factor F is the factor of safety FS. For Janbu's method, the force E_i is assumed to act at the lower middle third of the ith side while N_i acts at the middle of the ith base.

From Table II, it is found that the value of K_c and FS are compatible and reasonable for all the methods except method-4. The value of FS given by method-4 is the maximum among all the five methods, but the corresponding value of K_c is not. Since the shear force among slices is neglected, Bishop's method gives a geotechnically unstable result. The Bishop's method was intended to apply to a circular failure surface, therefore, it is not surprising it does not do well in this example.

The F–K'_c curve obtained in computing FS is shown in Figure 3. Hoek[4] has discovered that, when method-3 is used, the value of K'_c may become numerically unstable for improper values of F. The phenomenon is discovered in the present computation for the above five methods. When

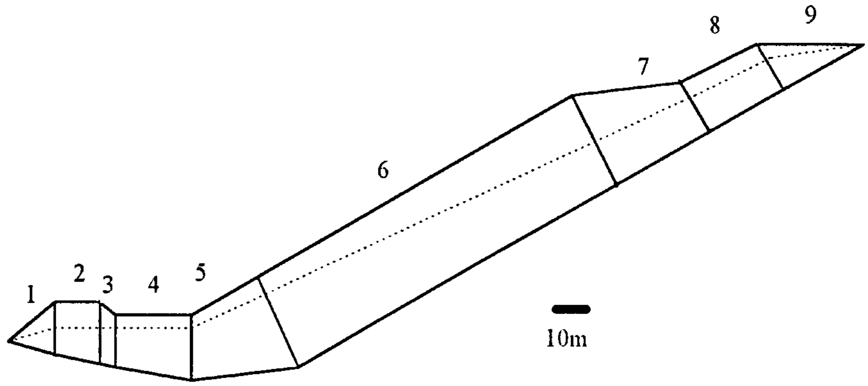

Figure 2. Slice partition of an open-pit coal mine slope[4]

Table I. Parameters of an open-pit coal mine slope and the computed results of Hoek's[4], Unit weight of water = 1, rock unit weight = 2·10

	Cohesion $c_{Bi} = 2{\cdot}00$					Friction angle $\varphi_{Bi} = 30{\cdot}00$ Force $TH_i = 0{\cdot}00$ $TV_i = 0{\cdot}00$				
Side number	1	2	3	4	5	6	7	8	9	10
Co-ordinate x_{Ti}	4·00	17·00	29·00	30·00	50·00	68·00	140·00	165·00	178·00	204·00
Co-ordinate y_{Ti}	17·00	26·00	26·00	24·00	25·00	37·00	88·00	90·00	99·00	103·00
Co-ordinate x_{Wi}	4·00	17·00	29·00	30·00	50·00	70·00	146·00	166·00	180·00	204·00
Co-ordinate y_{Wi}	17·00	23·00	22·00	22·00	24·00	33·00	80·00	89·00	96·00	103·00
Co-ordinate x_{Bi}	4·00	17·00	29·00	30·00	50·00	80·00	155·00	173·00	186·00	204·00
Co-ordinate y_{Bi}	17·00	12·00	10·00	10·00	8·00	11·00	65·00	80·00	89·00	103·00
Friction angle φ_{Si}	0·00	30·00	30·00	30·00	30·00	18·00	18·00	18·00	18·00	0·00
Cohesion c_{Si}	0·00	2·00	2·00	2·00	2·00	0·00	0·00	0·00	0·00	0·00
Sarma's acceleration $K_c = 0{\cdot}1008$										
Factor of safety FS = 1·17										

F_m ($m = 1, 2, 3, 4$ and 5, which correspond to the five methods) is less than a certain value F_{0m}, the value of K'_{cm} is oscillatory and sensitive to F_m, and the F_m–K'_{cm} curve has several branches. When $K'_{cm} = 0$, F_m may take multiple values. Hoek has pointed out that among these values only the one in the numerically stable domain of K'_{cm} is significant, e.g. it should be greater than F_{0m}. Thus only the branches of F_m–K'_{cm} curves with $F_m > F_{0m}$ are drawn in Figure 3. The lines $F_m = F_{0m}$ are, respectively, the asymptotes of these curves. In the example, for method-3, $F_{03} = 0{\cdot}8$, while for the other four methods, F_{0m} is in the range 0·3–0·4. The $1/F$–K'_c curves in the same domain as in Figure 3 are shown in Figure 4. They are all segmentally linear, illustrating that the F–K'_c curves are segmental hyperbolas.

Sarma and Bhave[2] have computed a great number of examples by method-1 and method-4, discovering that the factor of safety has approximately a linear relatinship with Sarma's acceleration:

$$FS = 1 + 3{\cdot}33K_c \tag{20}$$

Table II. Sarma's acceleration and factor of safety under different methods

	Method-1	Method-2	Method-3	Method-4	Method-5
K_c	− 0·0159	0·0099	0·1008	0·0787	0·0633
FS	0·9612	1·0239	1·1550	1·1782	1·1394

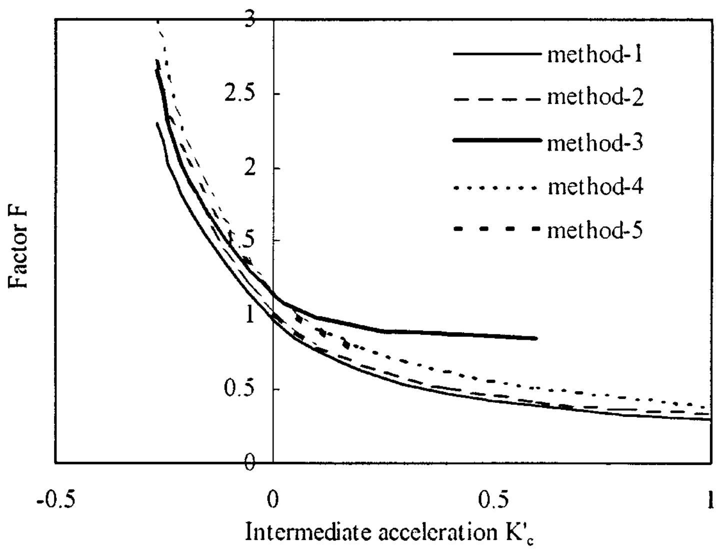

Figure 3. Curves of factor F versus intermediate Sarma' acceleration K'_c

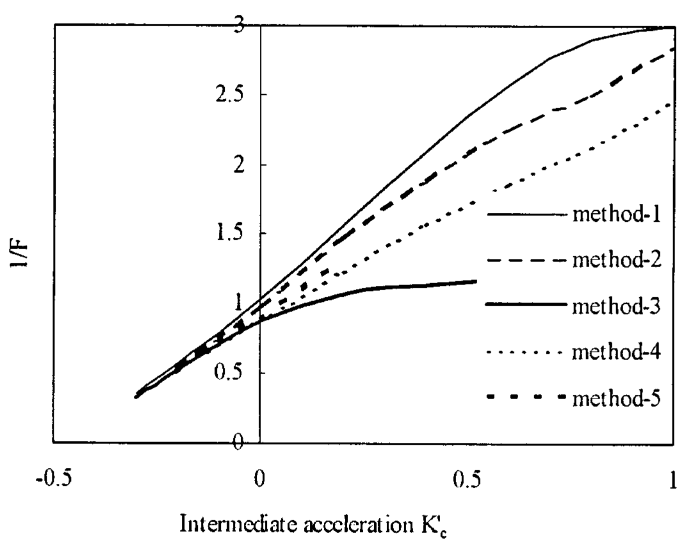

Figure 4. Curves of reciprocal factor $1/F$ versus intermediate Sarma' acceleraation K'_c

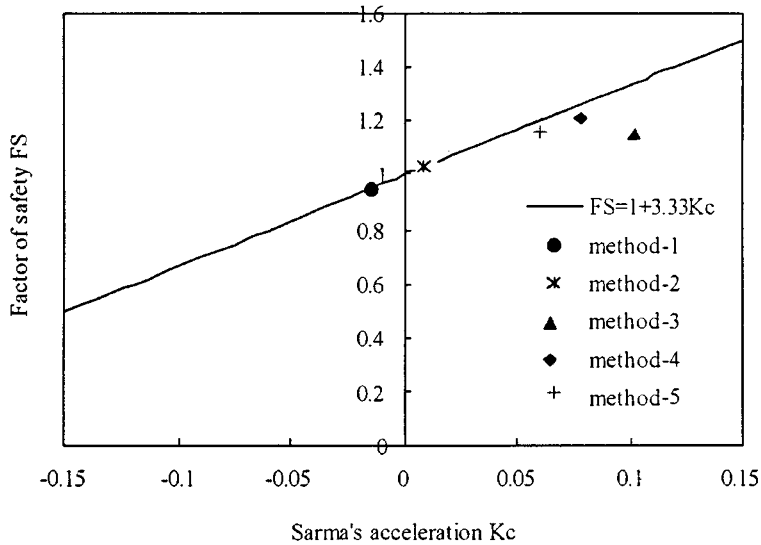

Figure 5. FS–K_c relationship

The presently computed values of (FS, K_c) under the five methods are compared with equation (20) in Figure 5. The results of method-1 and method-2 agree very well with equation (20), while those of method-4 and method-5 have some deviation. The result of method-3 appears not to satisfy equation (20).

The computation shows that, among all the five methods, method-1 is the most conservative, no matter which one of FS and K_c is taken as the measure of safety. It is interesting that method-4 is the least conservative when FS is taken as the measure of safety, however, when K_c is the measure, the least conservative is method-3.

In this paper, Cramer's rule is used since it can express Sarma's acceleration in a closed form, so that theoretical derivations can be clearly made. On the other hand, Cramer's rule is not an efficient linear equation solver. Therefore, it is not used directly in the computation. Instead, a more efficient solver is used to solve equations (5), so as to get the value of K_c. The distinction between Sarma's acceleration K_c and the intermediate Sarma's acceleration K'_c needs to be noted.

6. CONCLUSIONS

Using arbitrary slices, we employ Sarma's method, which is generally extended by modifying the key constraint equations, to analyse two-dimensional slopes. For five commonly used slice methods, here called method-1–method-5, a general closed-form expression of Sarma's acceleration is given by Cramer's rule. Furthermore, for the first four methods listed above, an analytical solution of Sarma's acceleration is derived. The present slope stability computation is convenient, for it can include several slice methods in a single program. It reveals reasonable and significant relationships among the five methods.

ACKNOWLEDGEMENTS

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APPENDIX I

I.1. Elements of the first 3n lines of the matrix A and the vector C

Elements of the first $3n$ lines of the matrix A are:

$a_{11} = -\cos\delta_2, a_{1,n} = -\sin\delta_2; a_{1,(2n-1)} = -\sin a_1; a_{1,(3n-1)} = \cos a_1; a_{1,(4n-1)} = -W_1$

$a_{1,j} = 0, (2 \leqslant j \leqslant 4n-2; j \neq n, 2n-1, 3n-1.)$

$a_{21} = \sin\delta_2; a_{2,n} = -\cos\delta_2; a_{2,(2n-1)} = \cos\alpha_1; a_{2,(3n-1)} \sim \sin\alpha_1$

$a_{2,j} = 0, \quad (2 \leqslant j \leqslant 4n-1; j \neq n, 2n-1, 3n-1)$

$a_{(2i-1),(i-1)} = \cos\delta_i; a_{(2i-1),i} = -\cos\delta_{i+1}; a_{(2i-1),(n+i-2)} = \sin\delta_i (i = 2, 3, \ldots, n-1)$

$a_{(2i-1),(n+i-1)} = -\sin\delta_{i+1}; a_{(2i-1)(2n+i-2)} = -\sin\alpha_i; a_{(2i-1),(3n+i-2)} = \cos\alpha_i$

$a_{(2i-1),(4n-1)} = -W_i$

$a_{(2i-1),j} = 0 \quad (j \neq i-1, i, n+i-2, n+i-1, 2n+i-1, 3n+i-2 \ \& \ 1 \leqslant j \leqslant 4n-2)$

$a_{2i,(i-1)} = -\sin\delta_i; a_{2i,i} = \sin\delta_{i+1}; a_{2i,(n+i-2)} = \cos\delta_i$

$a_{2i,(n+i-1)} = -\cos\delta_{i+1}; a_{2i,(2n+i-2)} = \cos\alpha_i; a_{2i,(3n+i-2)} = \sin\alpha_i$

$a_{2i,j} = 0, (j \neq i-1, i, n+i-2, n+i-1, 2n+i-2)$

$a_{(2n-1),(n-1)} = \cos\delta_n; a_{(2n-1),(2n-2)} = \sin\delta_n; a_{(2n-1),(3n-2)}$

$\quad = -\sin\alpha_n; a_{(2n-1),(4n-2)} = \cos\alpha_n; a_{(2n-1),(4n-1)} = -W_n$

$a_{(2n-1),j} = 0 \quad (1 \leqslant j \leqslant 4n-3; \quad j \neq n-1, 2n-2, 3n-2.)$

$a_{2n,(n-1)} = -\sin\delta_n; a_{2n,(2n-2)} = \cos\delta_n, a_{2n,(3n-2)} = \cos\alpha_n, a_{2n,(4n-2)} = \sin\alpha_n$

$a_{2n,j} = 0 \quad (1 \leqslant j \leqslant 4n-1 \ \& \ j \neq n-1, 2n-2, 3n-2, 4n-2)$

$a_{(2n+i),(2n+i-2)} = \tan\varphi_{Bi}; a_{(2n+i),(3n+i-2)} = -1$

$a_{(2n+1),i} = 0 \quad (j \neq 2n+i-2, 3n+i-2; 1 \leqslant j \leqslant 4n-1)$

Elements of the first $3n$ lines of the vector C are:

$$c_{(2i-1)} = -\mathrm{TH}_i, c_{2i} = \mathrm{TV}_i + W_i$$

$$c_{(2n+i)} = U_i \tan\varphi_{Bi} - c_{Bi} b_i / \cos\alpha_i (i = 1, 2, \ldots, n)$$

APPENDIX II

II.1. Deviation of equation (15)

Suppose $\mathrm{D}X''_i = X''_{i+1} - X''_i, \mathrm{D}E''_i = E''_{i+1} - E''_i$, where

$$\begin{aligned} X''_i &= X_i \cos\delta_i - E_i \sin\delta_i, \\ E''_i &= X_i \sin\delta_i + E_i \cos\delta_i \end{aligned} \quad (i = 2, \ldots, n) \tag{21}$$

The force equilibrium equations of the ith slice are

$$\mathrm{TS}_i \cos\alpha_i - N_i \sin\alpha_i = \mathrm{D}E_i'' + (K_c W_i - \mathrm{TH}_i) \tag{22}$$

$$\mathrm{TS}_i \sin\alpha_i + N_i \cos\alpha_i = \mathrm{D}X_i'' + (\mathrm{TV}_i + W_i)$$

It is solved from equation (22) that

$$\mathrm{TS}_i = [\mathrm{D}E_i'' + (K_c W_i - \mathrm{TH}_i)]\cos\alpha_i + [\mathrm{D}X_i'' + (\mathrm{TV}_i + W_i)]\sin\alpha_i \tag{23}$$

$$N_i = [\mathrm{D}X_i'' + (\mathrm{TV}_i + W_i)]\cos\alpha_i - [\mathrm{D}E_i'' + (K_c W_i - \mathrm{TH}_i)]\sin\alpha_i$$

It is given by substituting equation (23) into the limiting condition of the ith slice base that

$$\mathrm{D}X_i'' \tan(\varphi_{\mathrm{B}i} - \alpha_i) - \mathrm{D}E_i'' = P_i + K_c W_i \tag{24}$$

where

$$P_i = (U_i \sin\varphi_{\mathrm{B}i} - (c_{\mathrm{B}i} b_i \cos\varphi_{\mathrm{B}i} \cdot \sec\alpha_i)\sec(\varphi_{\mathrm{B}i} - \alpha_i) - (\mathrm{TV}_i + W_i)\tan(\varphi_{\mathrm{B}i} - \alpha_i) - \mathrm{TH}_i$$

Let $\mathrm{D}X_i = X_{i+1} - X_i = \lambda(F_{i+1} - F_i) = \lambda\,\mathrm{D}F_i$, where $X_i = \lambda F$, $\Sigma \mathrm{D}F_i = 0$. This is Morganstern's method of assuming shear force distribution among slices.[6] From equations (21), $\mathrm{D}X_i = \mathrm{D}X_i'' \cos\delta_i + \mathrm{D}E_i'' \sin\delta_i$. So

$$\mathrm{D}X_i'' \cos\delta_i + \mathrm{D}E_i'' \sin\delta_i = \lambda\,\mathrm{D}F_i \tag{25}$$

From equations (24) and (25),

$$\mathrm{D}X_i'' = (\lambda \mathrm{D}F_i + K_c W_i \sin\delta_i + P_i \sin\delta_i)\cos(\varphi_{\mathrm{B}i} - \alpha_i)\sec(\varphi_{\mathrm{B}i} - \alpha_i - \delta_i) \tag{26}$$

The centre of gravity of the slope, (x_G, y_G), is taken as the moment centre and the force N_i is assumed to act at the middle of the ith base. The moment equilibrium equation of the whole slope is

$$\sum(-\mathrm{TS}_i \cos\alpha_i + N_i \sin\alpha_i)(\bar{y}_i - y_G) + \sum(N_i \cos\alpha_i + \mathrm{TS}_i \sin\alpha_i)(\bar{x}_i - x_G)$$
$$- \sum \mathrm{TH}_i(y_i - y_G) - \sum \mathrm{TV}_i(x_i - x_G) = 0 \tag{27}$$

where $\bar{x}_i = \frac{1}{2}(x_{\mathrm{B}_{i+1}} + x_{\mathrm{B}_i})$, $\bar{y}_i = \frac{1}{2}(y_{\mathrm{B}_{i+1}} + y_{\mathrm{B}_i})$. It is obtained by substituting equations (22), (24) and (26) into equation (27) that

$$\lambda = \frac{V_2}{V_1} - \frac{V_3}{V_1} K_c \tag{28}$$

where V_1, V_2 and V_3 are given in equation (15). Thus it is equivalent to let in equation (11) that

$$\beta_{i+1} = 0,\ D_{i+1} = \frac{V_2}{V_1} F_{i+1},\ G_{i+1} = -\frac{V_3}{V_1} F_{i+1} \tag{29}$$

APPENDIX III

III.1. Derivation of equation (16)

In Figure 6, assume that within the ith slice all planes inclining at an angle α_i to the horizontal are in the state of limiting equilibrium. Sarma[1] has given that

$$\sigma_{xi} = d_{1i}\sigma_{yi} + d_{2i} + d_{3i}\Delta u_i \tag{30}$$

and

$$\sigma_{yi} = \gamma_r h_i' \cos\delta_i, \quad \Delta u_i = R_{ui}\gamma_r h_i' \cos\delta_i \tag{31}$$

where σ_{xi} and σ_{yi} are, respectively, stresses along the x- and y-axis; $R_{u_i} = U_i/(W_i \sec\alpha_i)$ is the pore pressure ratio;[1] $0 \leqslant h_i' \leqslant d_i$.

Rotate the co-ordinate axis clockwise with an angle δ_i. Similar to equation (30),

$$\sigma_{x_i^1} = d'_{1i}\sigma_{y_i^1} + d'_{2i} + d'_{3i}\Delta u_i \tag{32}$$

In equations (30) and (32), d_{1i}, d_{2i}, d_{3i}, d'_{1i}, d'_{2i} and d'_{3i} are given in equations (17). Besides,

$$\sigma_{x_i^1} + \sigma_{y_i^1} = \sigma_{x_i} + \sigma_{y_i} \tag{33}$$

From equataions (30), (32) and (33)

$$\sigma_{x_i^1} = \frac{1}{1 + d'_{1i}}\left[d'_{1i}(1 + d_{1i})\sigma_{yi} + (d'_{1i}d_{2i} + d'_{2i}) + (d'_{1i}d_{3i} + d'_{3i})\Delta u_i\right] \tag{34}$$

For a homogeneous slope, $E_i = \int_0^{d_i} \sigma_{x_i}\, dh'$. It is derived that $E_i = (\gamma_r d_i^2/2)\cos\delta_i K_i'$ by using equation (34) and equation (31), where K_i' is given in equations (17). Also $\mathrm{PW}_i = \frac{1}{2}R_{ui}\gamma_r d_i^2 \cos\delta_i$. So the shear force is

$$X_i = \left[(K_i' - R_{\mathrm{u}i})\frac{\gamma_r d_i^2}{2}\cos\delta_i \tan\varphi_{si} + c_{si}d_i\right]\frac{1}{\mathrm{FS}_{\mathrm{L}i}} \tag{35}$$

where $\mathrm{FS}_{\mathrm{L}i}$ is the local factor of safety of the ith side. Equation (16) is derived by letting $\mathrm{FS}_{\mathrm{L}i} = 1/(\lambda f(x))$ and assuming $f(x) = 1$.

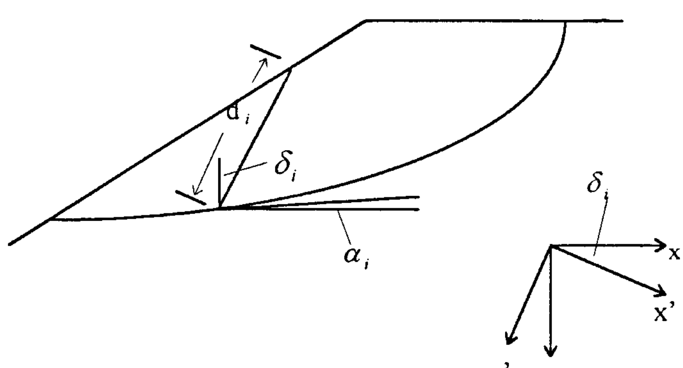

Figure 6. A homogeneous slope and the rotated co-ordinate axis

For a non-homogeneous slope, a similar method to that of Sarma's[1] may be used to perform the derivation.

NOTATIONS

c_{Si}, φ_{Si}	shear strength parameters on the ith slice side
c_{Bi}, φ_{Bi}	shear strength parameters on the ith slice base
d_i	length of the ith slice side
E_i, E_{i+1}	normal pressures on the ith and the $(i+1)$th slice side
F	factor dividing the shear strength parameters in the iterative computation of FS
FS	factor of safety
FS_{Li}	local factor of safety on the ith slice side
K_c	Sarma's acceleration
K'_c	intermediate Sarma's acceleration
l_i	location of acting point of N_i
N_i	normal pressure on the ith slice base
PW_i, PW_{i+1}	water force on the ith and the $(i+1)$th slice sides
TH_i	horizontal external force on the ith slice
TS_i	shear force on the ith slice base
TV_i	vertical external force on the ith slice
U_i	water force on the ith slice base
W_i	weight of the ith slice
X_i, X_{i+1}	shear forces on the ith and the $(i+1)$th slice sides
x_i, y_i	co-ordinates of the acting point of TH_i and TV_i
x_{Bi}, y_{Bi}	co-ordinates of the lower-left corner of the ith slice
x_{Ti}, y_{Ti}	co-ordinates of the upper-left corner of the ith slice
x_G, y_G	co-ordinates of the centre of gravity of the slope
x_{Gi}, y_{Gi}	co-ordinates of the centre of gravity of the ith slice
$\bar{x}_i$, $\bar{y}_i$	co-ordinates of the mid-point of the ith slice base
Z_i	location of acting point of E_i
α_i	angle made by the ith slice base with the x-axis
γ_r	unit weight of the slice material
δ_i	angle made by the ith slice side with the y-axis

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